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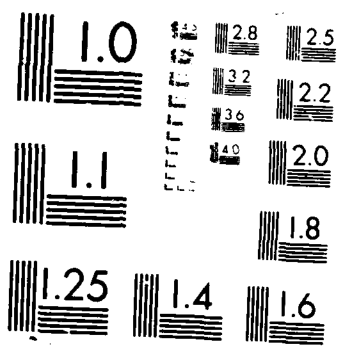

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AN ELEMENTARY APPROACH TO THE DANIELL-KOLMOGOROV THEOREM

AND SOME RELATED RESULTS

by

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AN ELEMENTARY APPROACH TO THE DANIELL-KOLMOGOROV THEOREM

AND SOME RELATED RESULTS

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Abstract: We give a short elementary proof of the Daniell-Kolmogorov existence theorem for probability measures on product spaces, assuming nothing but the existence of Lebesgue measure on the unit interval. Related approaches are used to prove the existence of regular conditional distributions directly on Polish spaces, and to establish the existence of random measures and sets with given finite-dimensional distributions or hitting probabilities, respectively.

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1. Introduction

Few results in probability theory are more fundamental or more well-known than the Daniell-Kolmogorov existence theorem (often attributed to Kolmogorov, though first proved by Daniell). It states that there exist random processes X_t, $t \in T$, with arbitrarily prescribed finite-dimensional distributions, subject only to the obvious consistency requirements. Here the index set T is completely arbitrary, but it is necessary to impose some restrictions on the state space $(S, \mathcal{B})$, and one usually assumes S to be Polish with $\mathcal{B}$ as the Borel σ-field. For further discussion, the reader may e.g. consult Billingsley (1986), who gives two detailed proofs in the case when S=R, and additional approaches in various special cases. An elegant but quite advanced discussion for more general spaces may be found in Dellacherie & Meyer (1975). See also Shiryayev (1985) for a (classical) counterexample to the statement for general S.

The standard textbook proofs are all rather advanced already for S=R, in requiring general results on the extension of measures, on the existence of regular conditional distributions, or on compactness in measure spaces. A further extension to arbitrary Polish spaces S requires the non-trivial fact that S can be embedded as a Borel subset into the real line. Our first aim in this paper is to give a simple elementary proof, which uses only the existence of Lebesgue measure on the unit interval. The latter seems unavoidable, since already the problem of assigning probabilities in the classical coin-tossing scheme ($X_1, X_2, \ldots$ i.i.d. with $P\{X_i = \pm 1\} = 1/2$) is equivalent to the construction of Lebesgue measure on [0,1], via the binary expansion of real numbers. (Cf. Section 1 in Billingsley (1986) for an extensive discussion of this point.) Our proof for S=R applies essentially without changes to arbitrary Polish spaces, so the extension step from R to a general S is eliminated. However, both separability and completeness seem to be essential for our approach, so other methods will be needed for a further extension to more general state spaces.

An equally fundamental and well-known result in probability theory is Doob's

theorem on the existence of regular conditional distributions. Recall that a version of a conditional probability function $P[X\in B|\mathcal{A}]$, $B\in\mathcal{B}$, is said to be regular, if every realization is a probability measure on S. As before one has to impose restrictions on the state space $(S,\mathcal{B})$ of the random element X, and again one usually assumes S to be Polish. The two theorems are closely related, in that the existence of regular conditional distributions implies the existence of processes with given finite-dimensional distributions, via the Ionescu-Tulcea theorem (cf. Shiryayev (1985)). (In particular, any counterexample for the latter result will provide one even for the former. A direct counterexample for the former result with general S is given in Doob (1953).)

The usual proofs of Doob's theorem for S=R are not hard (cf. Billingsley (1986)), but an extension to general Polish state spaces will again require the non-trivial embedding argument mentioned before. Alternatively, one may proceed directly viz Riez' representation theorem, as in Dellacherie & Meyer (1975). In the present paper we shall give an elementary proof directly for Polish state spaces, by applying the Daniell-Kolmogorov theorem to the sample realizations of the conditional probability function. Our approach has the further advantage of extending rather easily to deal with the existence of general random measures.

To be more specific, recall that the distribution of a random measure ξ on S is determined by its finite-dimensional projections

$$p_{B_1,\ldots,B_n} = P(\xi B_1,\ldots,\xi B_n)^{-1}, \quad B_1,\ldots,B_n\in\mathcal{B}, \quad n\in N. \tag{1}$$

An obvious problem is then to impose conditions (in addition to consistency) on a family of finite-dimensional distributions $p_{B_1,\ldots,B_n}$ (possibly with the B_j restricted to some suitable subclass $\mathcal{B}'\subset\mathcal{B}$), ensuring the existence of some random measure ξ on S satisfying (1) (with $\mathcal{B}$ replaced by $\mathcal{B}'$). Results of this type have been given by many authors, including Nawrotzki (1962), Harris (1968), Matthes, Kerstan & Mecke (1974-78), Ripley (1976), and Mecke (1979), but our approach in this paper may be easier and more elementary. It is somewhat related to the weak convergence approach in Kallenberg (1975-86), though the latter depends

on Prohorov's theorem, and on related compactness criteria in measure spaces.

There is a related problem for <u>random point fields</u> or <u>discrete random sets</u> φ in S (often called <u>simple point processes</u>, even for S=R). Here <u>discrete</u> means that all points of φ are isolated, and it is further assumed that $\varphi B = 1\{\varphi \cap B \neq \emptyset\}$ be measurable for all $B \in \mathcal{B}$. By a simple monotone class argument, it is seen that the distribution of φ is determined by the set of <u>avoidance probabilities</u>

$$T_B = P\{\varphi B = 0\}, \qquad B \in \mathcal{B}, \tag{2}$$

and the existence problem is then to impose conditions on a function T_B (possibly again with B restricted to some subclass $\mathcal{B}' \subset \mathcal{B}$), such that a random point field φ will exist satisfying (2). Even this problem has been discussed extensively in the literature, and some different approaches may be found in Kurtz (1974), Matthes et al. (1974-78), Kallenberg (1975-86), and Ripley (1976).

In the present paper we shall consider two new approaches to this problem. The first one is based on the existence theorem for random measures, mentioned earlier. The idea is then to construct, under suitable hypotheses on T, some random measure ξ on S satisfying

$$P\{\xi B = 0\} = T_B, \qquad B \in \mathcal{B}'. \tag{3}$$

If we can show in addition that ξ has discrete support φ, then this φ may be taken as our random point field.

Our second approach is based on an existence criterion for general random sets. Here the problem is to find conditions on a function T_B to ensure the existence of some closed random set φ satisfying (2). The basic result is due to Choquet (1953), who characterizes the permissible functions as alternating and suitably normalized capacities on S, and we may refer to Kendall (1974), Matheron (1975), and Norberg (1984) for extensive discussions and further results. Usually one needs to assume that S is locally compact and second countable. In the present paper, however, we shall use an elementary approach to obtain a similar result for arbitrary Polish spaces. We shall also show how further conditions may be added on T, to ensure that the random set φ in [illegible] discrete (and hence a random point field) or perfect.

Our program for the subsequent sections is first to discuss the Daniell-Kolmogorov theorem in Section 2, and then to turn to the existence of random measures and sets in Sections 3 and 4 respectively. Unless otherwise stated, all random elements below are assumed to be defined on some fixed probability space $(\Omega, \mathcal{A}, P)$ with generic points ω. We shall further assume the state space S to be Polish with Borel σ-field $\mathcal{B}$, and with classes $\mathcal{G}$ and $\mathcal{F}$ of open and closed sets. The interior, closure, boundary and complement of a set B are denoted by B^{o}, B^{-}, ∂B and B^{c}. We shall say that a class $\mathcal{C} \subset \mathcal{B}$ is <u>separating</u>, if for any $s \in G \in \mathcal{G}$ there exists some set $C \in \mathcal{C}$ with $s \in C^{o} \subset C \subset G$. In this case there will clearly exist a countable subclass with the same property, and it is further seen that every $G \in \mathcal{G}$ is a union of sets in $\mathcal{C}$. If S is locally compact, then the finite unions of sets in $\mathcal{C}$ will form a separating class in the sense of Norberg (1984), while the ring or semiring generated by $\mathcal{C}$ will be a <u>DC-ring</u> or <u>DC-semiring</u> in the sense of Kallenberg (1975-86).

Distances between points and diameters of sets in S are throughout defined in terms of some fixed metric d. The sets $B_{nj} \in \mathcal{B}$, $n, j \in N$, are said to form a <u>null-array of partitions of</u> S, if the B_{nj} form a partition of S for every fixed n into disjoint non-empty subsets, in such a way that every set $B_{n+1,j}$ is a subset of some B_{ni}, and such that the diameters of B_{nj} tend uniformly (in j) to zero as $n \to \infty$.

2. Measures on product spaces

The Daniell-Kolmogorov theorem is almost too well-known to require a formal statement. Let π_n denote the projection of S^m (for $m \geq n$) or S^∞ onto the subspace of the first n coordinates, and say that a sequence of measures μ_n on S^n, $n \in N$, is <u>projective</u>, if $\mu_m \pi_n^{-1} = \mu_n$ for all $m \geq n \geq 1$. A <u>projective limit</u> of the μ_n is a measure μ on S^∞ satisfying $\mu \pi_n^{-1} = \mu_n$ for all n. We assume all measures to be defined on the respective product σ-fields $\mathcal{B}^n$ and $\mathcal{B}^\infty$, which in our case coincide with the Borel σ-fields in S^n and S^∞ endowed with the product topologies.

<u>Theorem 2.1</u> (Daniell). <u>Let</u> S <u>be Polish. Then every projective sequence of probability measures</u> μ_n <u>on</u> S^n, $n \in N$, <u>has a projective limit</u> μ <u>on</u> S^∞.

Recall for later reference that the result extends immediately to the case of arbitrary index sets T (cf. Billingsley (1986)). Thus a family of probability measures μ_J with $J \subset T$ finite, which is <u>projective</u> in the sense that $\mu_K \pi_J^{-1} = \mu_J$ whenever $J \subset K$, has a projective limit μ on S^T satisfying $\mu \pi_J^{-1} = \mu_J$ for all J. Here π_J denotes the natural projection of S^K onto S^J, defined whenever $K \supset J$.

The idea of our proof is most transparent (at least to probabilists) when phrased in terms of random variables and their equality or convergence in distribution (denoted by $\overset{d}{=}$ or $\xrightarrow{d}$, respectively), so we shall first outline a probabilistic proof in the case S=R, and then give a detailed non-probabilistic version of the argument for general Polish state spaces.

<u>Proof for S=R.</u> For each $n \in N$, let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be a random vector with distribution μ_n, and note that by hypothesis

$$(X_1^{(m)}, \ldots, X_n^{(m)}) \overset{d}{=} (X_1^{(n)}, \ldots, X_n^{(n)}), \quad m \geq n.$$

We need to construct some random variables $X_1, X_2, \ldots$ on a suitable probability space, such that

$$(X_1, \ldots, X_n) \overset{d}{=} (X_1^{(n)}, \ldots, X_n^{(n)}), \quad n \in N. \tag{1}$$

If the $X_j^{(n)}$ are simple, we can easily construct $X_1, X_2, \ldots$ as simple step functions on the Lebesgue unit interval $I=[0,1)$, by successive partitions [illegible]

subintervals. (The details are spelled out in two special cases by Billingsley (1986), Theorems 5.2 and 8.1; the general case is similar.)

For general distributions, approximate each $X_j^{(n)}$ in the usual way by a monotone sequence of simple random variables $X_{kj}^{(n)}$ (cf. Theorem 13.5 in Billingsley (1986)). Then clearly

$$(X_{kj}^{(m)},\ k\in N,\ j=1,\ldots,n) \overset{d}{=} (X_{kj}^{(n)},\ k\in N,\ j=1,\ldots,n), \quad m\geq n,$$

so by ordering the pairs (k,j) in a sequence, it is seen from the argument in the special case that there exist some simple random variables X_{kj}, $k,j\in N$, defined as step functions on I, such that

$$(X_{kj},\ k\in N,\ j=1,\ldots,n) \overset{d}{=} (X_{kj}^{(n)},\ k\in N,\ j=1,\ldots,n), \quad n\in N.$$

Since the $X_{kj}^{(n)}$ are monotone in k for fixed j and n, the same thing must be true for the X_{kj} with j fixed, so the limits X_j must exist on the extended real line, and we get as $k\to\infty$

$$(X_1,\ldots,X_n) \overset{d}{\leftarrow} (X_{k1},\ldots,X_{kn}) \overset{d}{=} (X_{k1}^{(n)},\ldots,X_{kn}^{(n)}) \overset{d}{\rightarrow} (X_1^{(n)},\ldots,X_n^{(n)}).$$

Thus the X_j are a.s. finite and satisfy (1). □

<u>Proof for arbitrary Polish S.</u> Let us first assume that the μ_n have countable supports, and write

$$D = \bigcup_{n=1}^{\infty}\{s\in S:\ \mu_n(S^{n-1}\times\{s\})>0\},$$

so that μ_n is supported by D^n for each n. For notational convenience, identify D with N, so that the projective property becomes

$$\mu_n(r) = \sum_{k=1}^{\infty} \mu_{n+1}(r,k), \quad r\in N^n. \tag{2}$$

Construct a step function h_1 on the unit interval $I=[0,1)$, by dividing I into right-closed intervals I_k of length $\mu_1(k)$, starting from the left, and defining $h_1(x)=k$ when $x\in I_k$. Given that $H_n=(h_1,\ldots,h_n)$ has been constructed such that $H_n(x)=r\in N^n$ on some interval I_r of length $\mu_n(r)$, we may proceed to construct h_{n+1} on I_r by a partitioning into right-closed subintervals $I_{r,k}$ of length $\mu_{n+1}(r,k)$, and by putting $h_{n+1}(x)=k$ on $I_{r,k}$. Note that the construction is possible by (2), and that H_n maps Lebesgue measure λ into μ_n for each n. The entire sequence

$H=(h_1,h_2,\dots)$ is a measurable mapping from I to N^∞, and it is easily seen that the induced measure $\mu=\lambda H^{-1}$ has the desired properties.

In the general case, let $\{B_{kj}\}\subset\mathcal{B}$ be a null-array of partitions of S, fix arbitrary points $b_{kj}\in B_{kj}$, and define the mappings $g_1,g_2,\dots: S\to S$ by

$$g_k(s) = b_{kj} \quad \text{when} \quad s\in B_{kj}, \qquad k,j\in N.$$

For each $r=(s_1,\dots,s_n)\in S^n$, let $G_n(r)$ denote that array $g_k(s_j)$, $j=1,\dots,n$, $k\in N$. Then

$$\pi_n\circ G_m = G_n\circ\pi_n \quad \text{on} \quad S^m, \qquad m\geq n,$$

where the projection on the left is in index j, so we get

$$\mu_m G_m^{-1}\pi_n^{-1} = \mu_m\pi_n^{-1}G_n^{-1} = \mu_n G_n^{-1}, \qquad m\geq n.$$

Without ambiguity, we may hence define some measures ν_J on S^J with $J\subset N^2$ finite by

$$\nu_J = \mu_n G_n^{-1}\pi_J^{-1}, \qquad J\subset\{1,\dots,n\}\times N, \qquad n\in N. \tag{3}$$

Note that the ν_J have countable supports and satisfy

$$\nu_K\pi_J^{-1} = \nu_J, \qquad J\subset K\subset N^2 \text{ finite}.$$

By the first part of the proof, there must then exist some measure ν on S^{N^2}, such that

$$\nu_J = \nu\pi_J^{-1}, \qquad J\subset N^2 \text{ finite}. \tag{4}$$

(This can be seen most easily if we order the index set N^2 into a sequence, write J_n for the first n indices, and note that the measures ν_{J_n} form a projective sequence.) By (3) and (4) we have

$$\mu_n G_n^{-1}\pi_J^{-1} = \nu\pi_n^{-1}\pi_J^{-1}, \qquad J\subset\{1,\dots,n\}\times N, \qquad n\in N,$$

which means that

$$\mu_n G_n^{-1} = \nu\pi_n^{-1}, \qquad n\in N. \tag{5}$$

Now recall that $g_k(s)\to s$ for each $s\in S$, by construction. Thus (5) shows that, for ν-almost every array $r=(s_{kj})\in S^{N^2}$, the elements s_{kj} form Cauchy sequences in k for every j, so the limits $h_j(r)$ must exist. On the exceptional ν-nullset we may e.g. put $h_j(r)=b_{11}$. To see that the h_j are measurable, note that $f\circ h_j$ is trivially measurable for continuous $f: S\to R$, and conclude by approximation that $1_B\circ h_j$ is measurable for any open set $B\in\mathcal{B}$, and hence in general. Then

$H=(h_1,h_2,\ldots)$ is measurable $S^{N^2}\to S^N$, so we may define a measure $\mu=\nu H^{-1}$ on S^∞. By (5) we get for bounded continuous functions $f\colon S^n\to R$

$$\int f\circ(G_{k1},\ldots,G_{kn})\,d\mu_n = \int f\circ(\pi_{k1},\ldots,\pi_{kn})\,d\nu,$$

so by dominated convergence

$$\int f d\mu_n = \int f\circ\pi_n\circ H\, d\nu = \int f\circ\pi_n\, d\mu = \int f d(\mu\pi_n^{-1}).$$

By approximation and dominated convergence, this extends to indicators of open sets in S^n, and a monotone class argument then shows that $\mu_n=\mu\pi_n^{-1}$. Thus μ is the desired projective limit. $\square$

3. Conditional distributions and random measures

Given some measurable space $(S,\mathcal{B})$, a random (probability) measure on S, defined on some probability space $(\Omega,\mathcal{C},P)$, is a mapping $\xi:\mathcal{B}\times\Omega\to\overline{R}_+$, such that $\xi(B,\omega) = \xi B(\omega)$ is a (probability) measure in $B\in\mathcal{B}$ for fixed $\omega\in\Omega$ and a random variable in ω for fixed B. If X is a random element in S while $\mathcal{A}$ is a sub-σ-field of $\mathcal{C}$, then a regular version of the conditional probability function $P[X\in\cdot\,|\mathcal{A}]$ is a random probability measure ξ on S, which is $\mathcal{A}$-measurable and satisfies

$$P[X\in B|\mathcal{A}] = \xi B \quad \text{a.s.,} \qquad B\in\mathcal{B}. \tag{1}$$

Recall that, in this paper, S is Polish while $\mathcal{B}$ is the Borel σ-field in S.

Theorem 3.1 (Doob). *Let X be a random element in some Polish space S, and defined on some probability space $(\Omega,\mathcal{C},P)$ with sub-σ-field $\mathcal{A}$. Then the conditional probability function $P[X\in\cdot\,|\mathcal{A}]$ has a regular version.*

Proof. If X has countable range $C\subset S$, we may choose some versions

$$\eta_s = P[X=s|\mathcal{A}], \qquad s\in C,$$

with $\eta_s\geq 0$ for all s and $\sum\eta_s=1$, and define

$$\xi B = \sum_{s\in B\cap C}\eta_s, \qquad B\in\mathcal{B}.$$

Then ξ is both $\mathcal{A}$-measurable and measure valued, and moreover

$$P[X\in B|\mathcal{A}] = P[X\in B\cap C|\mathcal{A}] = \sum_{s\in B\cap C} P[X=s|\mathcal{A}] = \xi B \quad \text{a.s.,} \qquad B\in\mathcal{B},$$

so ξ is indeed a regular version of $P[X\in\cdot\,|\mathcal{A}]$.

In the general case, define functions $g_1, g_2, \ldots$ as in the proof of Theorem 2.1, and put $X_k\equiv g_k\circ X$ and $Y_n\equiv(X_1,\ldots,X_n)$. If $D=\{X_n(\omega);\,\omega\in\Omega,\ n\in N\}$, then each D^n is countable, so the conditional probability function $P[Y_n\in\cdot\,|\mathcal{A}]$ has a regular version μ_n with support in D^n, and we get in particular

$$\mu_{n+1}(\{r\}\times S) = P[Y_n=r|\mathcal{A}] = \mu_n\{r\} \quad \text{a.s.,} \qquad r\in D^n,\ n\in N. \tag{2}$$

Since only countably many conditions are involved, we may assume that even (2) holds identically. In that case

$$\mu_{n+1}(B\times S) = \sum_{r\in B\cap D^n}\mu_{n+1}(\{r\}\times S) = \sum_{r\in B\cap D^n}\mu_n\{r\} = \mu_n B, \qquad B\in\mathcal{B}^n,\ n\in N,$$

so every realization of the sequence (μ_n) is projective, and therefore

a projective limit $\mu=\mu(\omega)$ must exist for every $\omega\in\Omega$, by Theorem 2.1 above. (Note that only the countable case is needed here.)

Now assume the sets B_{nj} in the construction of g_n to be bounded in diameter by some quantities $\varepsilon_n \downarrow 0$, and write L for the class of convergent sequences in S^∞. Then clearly

$$\mu(L^c) \leq \mu\bigcup_{m\geq n}\{s\in S\ ;\ d(s_m,s_n)>\varepsilon_n\} \leq \sum_{m\geq n}\mu_m\{s\in S^m;\ d(s_m,s_n)>\varepsilon_n\}$$

$$= \sum_{m\geq n} P[d(X_m,X_n)>\varepsilon_n|\mathcal{A}] = 0 \quad \text{a.s.},$$

so by modifying the measures μ_n as well as μ on an $\mathcal{A}$-nullset, we may assume that $\mu(L)=1$. We now define $h(r)=\lim s_n$ for $r=(s_n)\in L$, and put $h(r)=b_{11}$ on L^c. Then h is clearly measurable $S^\infty\to S$, so we may put $\xi=\mu h^{-1}$. We claim that ξ is a regular version of $P[X\in\cdot|\mathcal{A}]$.

To see this, note that for bounded continuous functions $f: S\to R$

$$\int f(s_n)\mu(dr) = \int f(s_n)\mu_n(dr) = E[f\circ X_n|\mathcal{A}] \quad \text{a.s.}, \qquad n\in N.$$

Letting $n\to\infty$, we get by dominated convergence on each side

$$\int fd\xi = \int fd(\mu h^{-1}) = \int f\circ h\ d\mu = E[f\circ X|\mathcal{A}] \quad \text{a.s.}$$

By approximation we get the same result for indicators $f=1_G$ with $G\in\mathcal{G}$, and (1) then follows by a monotone class argument. Note also that the $\mathcal{A}$-measurability of the μ_n carries over to the integrals $\int fd\xi$, hence to all ξG with $G\in\mathcal{G}$, and finally to arbitrary ξB. □

Let us next consider the existence problem for general random measures. We are then looking for conditions on a projective family of probability measures p_J, with J a finite subset of $\mathcal{B}$ or of some suitable subring $\mathcal{U}$, in order that there should exist some random measure ξ on S with finite-dimensional distributions p_J on $\mathcal{B}$ or $\mathcal{U}$. Since the event that ξ be countably additive is not in the product σ-field (unless S is countable) it has to be replaced by the weaker requirement that

$$\xi B = \sum_{j=1}^{\infty}\xi B_j \quad \text{a.s.}, \qquad B_1,B_2,\ldots\in\mathcal{U} \text{ disjoint with union } B\in\mathcal{U}, \tag{3}$$

which is clearly equivalent to the two conditions

$$\xi(B \cup C) = \xi B + \xi C \quad \text{a.s.}, \qquad B, C \in \mathcal{U} \text{ disjoint}, \tag{4}$$

$$\xi B_n \xrightarrow{P} 0, \qquad B_1, B_2, \ldots \in \mathcal{U} \text{ with } B_n \downarrow \emptyset, \tag{5}$$

or in terms of the p_J,

$$p_{B,C,B\cup C}\{(x,y,z);\ x+y=z\} = 1, \qquad B, C \in \mathcal{U} \text{ disjoint}, \tag{6}$$

$$p_{B_n} \xrightarrow{w} \delta_0, \qquad B_1, B_2, \ldots \in \mathcal{U} \text{ with } B_n \downarrow \emptyset. \tag{7}$$

The existence theorems of Nawrotzki (1962), Harris (1968), and Matthes et al. (1974-78) state that, under suitable assumptions on $\mathcal{U}$, (6) and (7) are indeed sufficient for the existence of some random measure ξ as above.

The first step in the proof is typically to infer from the Daniell-Kolmogorov theorem that there exists some random process η on $\mathcal{U}$ with finite-dimensional distributions p_J, and hence satisfying (4) and (5). Since these conditions are equivalent to (3), we may just as well assume from the beginning that η is a random process on $\mathcal{U}$ satisfying

$$\eta(B) = \sum_{j=1}^{\infty} \eta(B_j) \quad \text{a.s.}, \qquad B_1, B_2, \ldots \in \mathcal{U} \text{ disjoint with union } B \in \mathcal{U}, \tag{8}$$

and then try to construct a measure valued version ξ of η. The theorem below is a version of the classical result, but the present approach may be easier and more elementary than previous ones.

Theorem 3.2. Let S be Polish with Borel-σ-field $\mathcal{B}$ and a separating ring $\mathcal{U} \subset \mathcal{B}$, and let η be an R_+-valued random process on $\mathcal{U}$ satisfying (8). Then there exists an a.s. unique locally finite random measure ξ on S satisfying

$$\xi B = \eta(B) \quad \text{a.s.}, \qquad B \in \mathcal{U}. \tag{9}$$

Proof. Divide S into subsets $C_1, C_2, \ldots \in \mathcal{U}$. If we can prove the existence of some random measures ζ_n satisfying

$$\zeta_n B = \eta(B \cap C_n) \quad \text{a.s.}, \qquad B \in \mathcal{U},\ n \in N,$$

then it is clear from (8) that the random measure $\xi = \sum \zeta_n$ satisfies (9). We may thus assume that $\eta(B \setminus C) = 0$ for some fixed set $C \in \mathcal{U}$ and for all $B \in \mathcal{U}$. Conditioning on the event $\{\eta(C) > 0\}$ and dividing by $\eta(C)$, we may further reduce to the case when $\eta(C) = 1$. In that case, we may proceed as in the proof of Theorem 3.1 to construct a null-array $\{B_{nj}\} \subset \mathcal{U}$ of partitions of S and an associated sequence

of discrete random measures ξ_n satisfying

$$\xi_n B_{nj} = \eta(B_{nj}) \quad \text{a.s.}, \qquad n, j \in N, \tag{10}$$

and converging weakly for each $\omega \in \Omega$ towards some random measure ξ.

To prove that ξ satisfies (9), let $\mathcal{U}'$ denote the ring generated by $\{B_{nj}\}$, and note that, by (8) and (10),

$$\eta(B) = \lim_{n\to\infty} \xi_n B \leq \limsup_{n\to\infty} \xi_n B^- \leq \xi B^- \quad \text{a.s.}, \qquad B \in \mathcal{U}'. \tag{11}$$

Next fix $B \in \mathcal{U}'$ and $G \in \mathcal{G}$ with $B \subset G$, and choose $B_1, B_2, \ldots \in \mathcal{U}'$ with $B_n^- \subset G$ and $B_n \uparrow G$. Then $B_n \cap B \uparrow B$, so by (8) and (11),

$$\xi G \geq \xi B_n^- \geq \eta(B_n) \geq \eta(B_n \cap B) \rightarrow \eta(B) \quad \text{a.s.},$$

which shows that

$$\eta(B) \leq \xi G \quad \text{a.s.}, \qquad B \in \mathcal{U}' \text{ and } G \in \mathcal{G} \text{ with } B \subset G. \tag{12}$$

By the regularity of the intensity measure $E\xi$, we may next choose, for fixed $B \in \mathcal{U}'$, some sets $G_n \in \mathcal{G}$ with $G_1 \supset G_2 \supset \ldots \supset B$ and $E\xi G_n \downarrow E\xi B$. Then $\xi G_n \downarrow \xi \bigcap G_n = \xi B$ a.s., so by (12)

$$\eta(B) \leq \xi B \quad \text{a.s.}, \qquad B \in \mathcal{U}'. \tag{13}$$

Assuming as we may that $C \in \mathcal{U}'$, we may apply (13) to $C \setminus B$ to obtain

$$\eta(B) = 1 - \eta(C \setminus B) \geq 1 - \xi(C \setminus B) = \xi(C \setminus B)^c \geq \xi B \quad \text{a.s.},$$

and by combining the two relations we get

$$\xi B = \eta(B) \quad \text{a.s.}, \qquad B \in \mathcal{U}'. \tag{14}$$

If ξ' is another random measure satisfying (14), then ξ and ξ' must agree on $\mathcal{U}'$ for ω outside some fixed nullset, so $\xi = \xi'$ a.s. by a monotone class argument. This proves the uniqueness assertion. To extend (14) to $\mathcal{U}$, fix an arbitrary set $U \in \mathcal{U}$, and repeat the above argument with partitioning sets $B_{nj} \cap U$ and $B_{nj} \setminus U$, to obtain a random measure ξ' satisfying (14) with $\mathcal{U}'$ augmented with the set U. Then $\xi' = \xi$ a.s. as above, and we get

$$\xi U = \xi' U = \eta(U) \quad \text{a.s.},$$

as desired. Finally note that, since the class $\{U^o;\ U \in \mathcal{U}\}$ contains a countable base, we can make ξ locally finite by changing its values on an Ω-nullset. $\square$

4. Random sets and point fields

Consider as before some Polish space S with closed and open sets $\mathcal{F}$ and $\mathcal{G}$, and with Borel σ-field $\mathcal{B}$. By a closed random set in S we mean a mapping φ of some probability space $(\Omega, \mathcal{O}, P)$ into $\mathcal{F}$, such that the function $\varphi G = 1\{\varphi \cap G \neq \emptyset\}$ is $\mathcal{O}$-measurable for every $G \in \mathcal{G}$. By the projection theorem (cf. Dellacherie & Meyer (1975)), φB is then universally measurable for every $B \in \mathcal{B}$.

A closed random set φ is called a random point field, if every realization of φ consists of only isolated points. A simple approximation argument then shows that the functions

$$\xi B = \#\{\varphi \cap B\}, \quad B \in \mathcal{B}, \tag{1}$$

are measurable, so that (1) defines a simple random measure ξ on S. (Here simple means purely atomic with atoms of unit size.) In particular, $\varphi B = 1\{\xi B > 0\}$ is seen to be measurable for every $B \in \mathcal{B}$. (Thus no completeness of $\mathcal{O}$ is needed in this case.) Conversely, every simple random measure ξ on S defines a unique random point field φ satisfying (1).

We have already noted that the distribution of a closed random set φ is determined by the function

$$T(B) = P\{\varphi B = 0\}, \quad B \in \mathcal{B}, \tag{2}$$

(even with B restricted to $\mathcal{G}$). Our aim is to look for conditions on T, in order that some random closed set of point field φ should exist satisfying (2). Since clearly

$$(-1)^n \Delta_{A_1} \cdots \Delta_{A_n} T(B) = P\{\varphi B = 0, \varphi A_1 = \cdots = \varphi A_n = 1\}, \tag{3}$$

where $\Delta_A T(B) = T(B \cup A) - T(B)$, a necessary condition is that T be alternating or completely monotone, in the sense that the left-hand side of (3) is non-negative for all $n \in Z_+$ and $A_1, \ldots, A_n, B \in \mathcal{B}$. Note also that necessarily $T(\emptyset) = 1$, since $\varphi\emptyset = 0$ identically for all φ. The following observation is essentially due to Choquet (1953), and may serve as a motivation for subsequent results.

Lemma 4.1. *Fix an arbitrary space* S, *and let* $\mathcal{U}$ *be a class of subsets of* S *which contains* $\emptyset$ *and is closed under finite unions. Assume that* $T: \mathcal{U} \to [0,1]$ *is alternating with* $T(\emptyset)=1$. *Then there exists some* $\{0,1\}$*-valued random process* ψ *on* $\mathcal{U}$ *satisfying* (3), *and such that moreover*

$$\psi(\emptyset) = 0 \quad \text{a.s.,} \tag{4}$$

$$\psi(B_1 \cup B_2) = \psi(B_1) \vee \psi(B_2) \quad \text{a.s.,} \qquad B_1, B_2 \in \mathcal{U}. \tag{5}$$

Conversely, any $\{0,1\}$*-valued process* ψ *on* $\mathcal{U}$ *satisfying* (4) *and* (5) *determines through* (2) *an alternating function* T *on* $\mathcal{U}$ *with* $T(\emptyset)=1$.

Proof. Given T as stated, construct as in (3) a family of finite-dimensional distributions on $\mathcal{U}$, and check that these are projective. By the Daniell-Kolmogorov theorem, there must then exist some process ψ satisfying (3), and it is easy to check that ψ will satisfy (4) and (5) as well. The converse statement is also easy to verify. □

We need additional conditions on T, in general, to ensure that ψ will have a set valued version (or more precisely, that $\psi(B) = \varphi B$ a.s. for some closed random set φ), and even more is needed if we want this version to be discrete, i.e. a random point field. As in the last section, we prefer to state our conditions directly in terms of the process ψ. From Lemma 4.1 it is clear how they could be rephrased in terms of T, if required.

As mentioned earlier, we shall discuss two different approaches to the existence problem for random point fields, leading to somewhat different results. Our first method uses the existence criterion for random measures in Theorem 3.2 above, and yields conditions similar to those of Karbe (cf. Matthes et al. (1974-78)) and Kurtz (1974). For convenience here and below, we shall write $\mathcal{N}_{B,\mathcal{U}}$ for the class of all finite collections of disjoint $\mathcal{U}$-sets included in the set B.

Theorem 4.2. Let S *be Polish with Borel* σ*-field* $\mathcal{B}$ *and a separating ring* $\mathcal{U} \subset \mathcal{B}$, *and let* ψ *be a* $\{0,1\}$*-valued random process on* $\mathcal{U}$, *satisfying*

(i) $\psi(B_1 \cup B_2) = \psi(B_1) \vee \psi(B_2)$ *a.s.*, $B_1, B_2 \in \mathcal{U}$,

(ii) $\psi(B_n) \xrightarrow{P} 0$ *a.s.*, $B_1, B_2, \ldots \in \mathcal{U}$ *with* $B_n \downarrow \emptyset$,

(iii) $\lim_{r\to\infty} \sup \{P\{\sum \psi(B_j) > r\};\ (B_j) \in \mathcal{M}_{B,\mathcal{U}}\} = 0, \quad B \in \mathcal{U}$.

Then there exists an a.s. unique random point field φ *in* S, *such that*

$$\varphi B = \psi(B) \quad \text{a.s.}, \qquad B \in \mathcal{U}. \tag{6}$$

Proof. Note first that (i) and (ii) imply

$$\psi(B) = \max \psi(B_j) \quad \text{a.s.}, \qquad B_1, B_2, \ldots \in \mathcal{U} \text{ with union } B \in \mathcal{U}. \tag{7}$$

Fix a null-array $\{B_{nj}\} \subset \mathcal{U}$ of partitions of S, and write $\mathcal{U}'$ for the generated ring. Define a mapping $\eta : \Omega \times \mathcal{U}' \to \bar{Z}_+$ by

$$\eta(B) = \limsup_{n\to\infty} \sum_{j=1}^{\infty} \psi(B_{nj} \cap B), \qquad B \in \mathcal{U}'. \tag{8}$$

Here the limit will in fact exist a.s. on the right, since the sum is a.s. non-decreasing in n by (7). It follows in particular that

$$\eta(B_1 \cup B_2) = \eta(B_1) + \eta(B_2) \quad \text{a.s.}, \qquad B_1, B_2 \in \mathcal{U}' \text{ disjoint}. \tag{9}$$

Moreover, it is seen from (iii) that, for $B \in \mathcal{U}'$ and $r \to \infty$,

$$P\{\eta(B) > r\} = \lim_{n\to\infty} P\Big\{\sum_{j=1}^{\infty} \psi(B_{nj} \cap B) > r\Big\} = \lim_{n\to\infty} \lim_{r\to\infty} P\Big\{\sum_{j=1}^{k} \psi(B_{nj} \cap B) > r\Big\} \to 0,$$

which shows that $\eta(B)$ is a.s. finite. Note also that, by (7) and (8),

$$\eta(B) \wedge 1 = \psi(B) \quad \text{a.s.}, \qquad B \in \mathcal{U}', \tag{10}$$

and conclude from (ii) that

$$\eta(B_n) \xrightarrow{P} 0, \qquad B_1, B_2, \ldots \in \mathcal{U}' \text{ with } B_n \downarrow \emptyset.$$

Combining this with (9) yields

$$\eta(B) = \sum_{j=1}^{\infty} \eta(B_j), \qquad B_1, B_2, \ldots \in \mathcal{U}' \text{ disjoint with union } B \in \mathcal{U}',$$

so Theorem 3.2 applies, and there must exist some locally finite random measure ξ on S satisfying

$$\xi B = \eta(B) \quad \text{a.s.}, \qquad B \in \mathcal{U}'. \tag{11}$$

A monotone class argument shows that ξ can be chosen to be $\bar{Z}_+$-valued. In that case,

its support φ must be discrete, and by (10) and (11)

$$\varphi B = \xi B \wedge 1 = \eta(B) \wedge 1 = \psi(B) \quad \text{a.s.}, \quad B \in \mathcal{U}'. \tag{12}$$

Another monotone class argument shows that φ is the a.s. unique random set satisfying (12), and so we may argue as in case of Theorem 3.2 to extend (12) to $\mathcal{U}$. □

We next prove a version of Choquet's (1953) existence theorem for closed random sets. A related criterion for locally compact spaces was obtained by Norberg (1984), who in turn refers to Wim Vervaat for similar (unpublished) results. Our approach is elementary and applies to arbitrary Polish spaces.

Theorem 4.3. Let S *be a Polish space equipped with a base* $\mathcal{V}$ *containing* $\emptyset$, *and let* ψ *be a* $\{0,1\}$*-valued random process on* $\mathcal{V}$ *satisfying*

(i) $\psi(\emptyset)=0$ *a.s.*,

(ii) $\psi(B) = \max \psi(B_j)$ *a.s.*, $B_1, B_2, \ldots \in \mathcal{V}$ *with union* $B \in \mathcal{V}$.

Then there exists an a.s. unique closed random set φ *in* S *satisfying*

$$\varphi B = \psi(B) \quad \text{a.s.}, \quad B \in \mathcal{V}. \tag{13}$$

Proof. Choose a countable base $\mathcal{V}' \subset \mathcal{V}$, and write $\mathcal{V}_n$ for the class of $\mathcal{V}'$-sets with diameter $< n^{-1}$. Since clearly

$$G = \bigcup\{C \in \mathcal{V}_n;\ C \subset G\} = \bigcup\{C \in \mathcal{V}';\ C^- \subset G\}, \quad G \in \mathcal{G},\ n \in N, \tag{14}$$

we get from (ii), outside a fixed Ω-nullset,

$$\psi(B) = \max\{\psi(C);\ C \in \mathcal{V}_n,\ C \subset B\}, \quad B \in \mathcal{V}',\ n \in N, \tag{15}$$

and we may modify ψ so that (i) and (15) hold identically. Next define a mapping $\chi : \Omega \times \mathcal{G} \to \{0,1\}$ by

$$\chi(G) = \max\{\psi(B);\ B \in \mathcal{V}',\ B^- \subset G\}, \quad G \in \mathcal{G}, \tag{16}$$

and note that, by (ii), (14) and (16),

$$\chi(B) = \psi(B) \quad \text{a.s.}, \quad B \in \mathcal{V}. \tag{17}$$

We shall prove that, for every $\omega \in \Omega$,

$$\chi(\bigcup\{G \in \mathcal{H}\}) = \max\{\chi(G);\ G \in \mathcal{H}\}, \quad \mathcal{H} \subset \mathcal{G}. \tag{18}$$

Here the inequality $\geq$ is obvious since $\chi(G)$ is non-decreasing in G, so we need only prove the relation $\leq$. Let us then fix $\omega \in \Omega$ and $\mathcal{H} \subset \mathcal{G}$, put $H = \bigcup \mathcal{H}$, and

assume that $\chi(H)=1$. Then there exists by (16) some $B\in\mathcal{V}'$ with $B^-\subset H$ and $\psi(B)=1$, and by (15) we may construct a sequence $B\supset B_1\supset B_2\supset\ldots$ with $B_n\in\mathcal{V}_n$ and $\psi(B_n)=1$. Since the B_n are non-empty by (i) while S is complete, the set $\bigcap B_n^-$ will consist of exactly one point $s\in B_n^-\subset H$. We may then choose some $G\in\mathcal{H}$ containing s, and hence containing B_n^- for large n. For such an n we get by (16), $\chi(G)\geq\psi(B_n)=1$, which shows that even the right-hand side of (18) equals 1.

Let us now define a mapping $\varphi:\Omega\rightarrow\mathcal{F}$ by

$$\varphi=\bigcap\{F\in\mathcal{F};\ \chi(F^c)=0\}.$$

Then $\chi(\varphi^c)=0$ by (18), so we get for arbitrary $G\in\mathcal{G}$

$$\varphi G=0\quad\Longleftrightarrow\quad G\subset\varphi^c\quad\Longleftrightarrow\quad\chi(G)=0,$$

which means that $\varphi G=\chi(G)$. Combining this with (17) yields (13). Note also that φG is measurable for each $G\in\mathcal{G}$ by (16), so that φ is indeed a random set. If φ' is another random closed set satisfying (13), then $\varphi' G=\varphi G$ holds outside a fixed nullset for all $G\in\mathcal{V}'$, and hence also for all $G\in\mathcal{G}$. Taking $G=\varphi^c$ or φ'^c, we get in particular $\varphi'\setminus\varphi=\varphi\setminus\varphi'=\emptyset$ a.s., so $\varphi'=\varphi$ a.s. Thus φ is a.s. unique. □

Our final aim is to show how the random set approach leads to a second set of conditions for the existence of a random point field. The same method leads without extra effort to an existence criterion in the other extreme case, namely for perfect random sets. Recall that a closed set is discrete if all its points are isolated, and perfect if none of them are. Recall also the definition of $\mathcal{M}_{B,\mathcal{U}}$, stated before Theorem 4.2 above.

Theorem 4.4. Let S be a Polish space equipped with a base $\mathcal{U}$ which is closed under finite unions, and let φ be a closed random set in S. Then φ is a.s. perfect if

$$\sup\{P\{\textstyle\sum\varphi B_j>1\};\ (B_j)\in\mathcal{M}_{B,\mathcal{U}}\}=P\{\varphi B=1\},\qquad B\in\mathcal{U},\tag{19}$$

while φ is a.s. discrete if

$$\lim_{r\to\infty}\sup\{P\{\textstyle\sum\varphi B_j>r\};\ (B_j)\in\mathcal{M}_{B,\mathcal{U}}\}=0,\qquad B\in\mathcal{U},\tag{20}$$

or if there exists some measure ν on S with

$$P\{\varphi B=1\}\leq\nu B<\infty,\qquad B\in\mathcal{U}.\tag{21}$$

Our proof will be based on two lemmas.

Lemma 4.5. Given φ, there exist random sets $\varphi_1 \subset \varphi_2 \subset \ldots \subset \varphi$ in S with $\#\varphi_n = \#\varphi \wedge n$ a.s. for all n.

Note that this follows immediately from the section theorem (cf. Dellacherie & Meyer (1975)), provided that $\mathbb{C}$ is complete. To keep the paper elementary, we sketch a simple direct proof.

Proof. Let $\mathcal{V}_n$ be such as in the last proof, and proceed as before to construct a sequence $B_1 \supset B_2 \supset \ldots$ with $B_n \in \mathcal{V}_n$, and such that $\varphi B_n = 1$ whenever $\varphi \neq \emptyset$. Then $\bigcap B_n^-$ consists of exactly one point σ_1, and it is easily seen that $\sigma_1 \in \varphi$ when $\varphi \neq \emptyset$. Since the $\mathcal{V}_n$ are countable, we may choose the 'first' permissible set in each step, to make sure that σ_1 will be measurable. Repeating the procedure with $\mathcal{V}_n$ replaced by $\mathcal{V}_n' = \{B \in \mathcal{V}_n;\ \sigma_1 \notin B\}$ yields a second point σ_2, and so forth. It is easily checked that the sets $\varphi_n = \{\sigma_j \in \varphi;\ j \leq n\}$ have the desired properties. □

Lemma 4.6. Let ξ be given by (1). Then

$$\sup\left\{P\{\textstyle\sum \varphi B_j \geq n\};\ (B_j) \in \mathcal{M}_{B,\mathcal{U}}\right\} = P\{\xi B \geq n\}, \qquad B \in \mathcal{U},\ n \in N.$$

Proof. The inequality $\leq$ is obvious since $\sum \varphi B_j \leq \xi B$. To prove the converse relation, we may assume by Lemma 4.5 that $\#\varphi = \#(\varphi \cap B) \leq n$. The measure $E\xi$ is then finite, so for each $k \in N$ we may choose a partition of B into Borel sets A_{kj} with diameter $< k^{-1}$, and such that $E\xi \partial A_{kj} = 0$. Note that

$$P\{\varphi A_{kj}^o \neq \varphi A_{kj}\} \leq P\{\varphi \partial A_{kj} = 1\} = P\{\xi \partial A_{kj} > 0\} \leq E\xi \partial A_{kj} = 0.$$

For each k and j we may next choose sets $B_{mkj} \in \mathcal{U}$ with $B_{mkj} \uparrow A_{kj}$. Then clearly

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} \varphi A_{kj} = \xi B,$$

while

$$\lim_{m\to\infty} \sum_{j=1}^{m} \varphi B_{mkj} = \sum_{j=1}^{\infty} \varphi A_{kj}^o = \sum_{j=1}^{\infty} \varphi A_{kj} \quad \text{a.s.},$$

so we get

$$\sup\left\{P\{\textstyle\sum \varphi B_j \geq n\};\ (B_j) \in \mathcal{M}_{B,\mathcal{U}}\right\} \geq \lim_{k\to\infty} \lim_{m\to\infty} P\left\{\sum_{j=1}^{m} \varphi B_{mkj} \geq n\right\}$$

$$= \lim_{k\to\infty} P\left\{\sum_{j=1}^{\infty} \varphi A_{kj} \geq n\right\} = P\{\xi B \geq n\},$$

as desired. □

<u>Proof of Theorem 4.4.</u> By Lemma 4.6, condition (19) is equivalent to

$$\xi B = \#(\varphi \cap B) \neq 1 \quad \text{a.s.}, \qquad B \in \mathcal{U}, \tag{22}$$

while (20) is equivalent to

$$\xi B = \#(\varphi \cap B) < \infty \quad \text{a.s.}, \qquad B \in \mathcal{U}, \tag{23}$$

and it is obvious that (22) is true when φ is a.s. perfect. To prove the sufficiency of the two conditions, we may assume that $\mathcal{U}$ is countable and throw away the exceptional nullsets. Then (22) is clearly impossible if φ has isolated points, while (23) is impossible if φ has accumulation points, so (22) implies that φ is perfect and (23) that it is discrete. Finally (21) implies (20), since if $(B_j) \in \mathcal{M}_{B,\mathcal{U}}$ and $r > 0$,

$$P\left\{\sum \varphi B_j > r\right\} \leq r^{-1} E \sum \varphi B_j \leq r^{-1} \sum \nu B_j \leq r^{-1} \nu B. \qquad \square$$

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